1 Vector Calculus

Below, \( \mathbf{x} \in \mathbb{R}^d \) means that \( \mathbf{x} \) is a \( d \times 1 \) (column) vector with real-valued entries. Likewise, \( \mathbf{A} \in \mathbb{R}^{d \times d} \) means that \( \mathbf{A} \) is a \( d \times d \) matrix with real-valued entries. In this course, we will by convention consider vectors to be column vectors.

Consider \( \mathbf{x}, \mathbf{w} \in \mathbb{R}^d \) and \( \mathbf{A} \in \mathbb{R}^{d \times d} \). In the following questions, \( \frac{\partial}{\partial \mathbf{x}} \) denotes the derivative with respect to \( \mathbf{x} \), while \( \nabla \) denote the gradient with respect to \( \mathbf{x} \). Compute the following:

(a) \( \frac{\partial \mathbf{w}^T \mathbf{x}}{\partial \mathbf{x}} \) and \( \nabla_{\mathbf{x}} (\mathbf{w}^T \mathbf{x}) \)

(b) \( \frac{\partial (\mathbf{w}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{x}} \) and \( \nabla_{\mathbf{x}} (\mathbf{w}^T \mathbf{A} \mathbf{x}) \)

(c) \( \frac{\partial (\mathbf{w}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{w}} \) and \( \nabla_{\mathbf{w}} (\mathbf{w}^T \mathbf{A} \mathbf{x}) \)

(d) \( \frac{\partial (\mathbf{w}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{A}} \) and \( \nabla_{\mathbf{A}} (\mathbf{w}^T \mathbf{A} \mathbf{x}) \)

(e) \( \frac{\partial (\mathbf{x}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{x}} \) and \( \nabla_{\mathbf{x}} (\mathbf{x}^T \mathbf{A} \mathbf{x}) \)

(f) \( \nabla_{\mathbf{x}}^2 (\mathbf{x}^T \mathbf{A} \mathbf{x}) \)

2 Eigenvalues

(a) Let \( \mathbf{A} \) be an invertible matrix. Show that if \( \mathbf{v} \) is an eigenvector of \( \mathbf{A} \) with eigenvalue \( \lambda \), then it is also an eigenvector of \( \mathbf{A}^{-1} \) with eigenvalue \( \lambda^{-1} \).

(b) A square and symmetric matrix \( \mathbf{A} \) is said to be positive semidefinite (PSD) (\( \mathbf{A} \succeq 0 \)) if \( \forall \mathbf{v} \neq 0, \mathbf{v}^T \mathbf{A} \mathbf{v} \geq 0 \). Show that \( \mathbf{A} \) is PSD if and only if all of its eigenvalues are nonnegative.

   Hint: Use the eigendecomposition of the matrix \( \mathbf{A} \).