CS 189 Introduction to Machine Learning
Spring 2018

DIS0

1 Vector Calculus

Below, $\mathbf{x} \in \mathbb{R}^d$ means that $\mathbf{x}$ is a $d \times 1$ (column) vector with real-valued entries. Likewise, $\mathbf{A} \in \mathbb{R}^{d\times d}$ means that $\mathbf{A}$ is a $d \times d$ matrix with real-valued entries. In this course, we will by convention consider vectors to be column vectors.

Consider $\mathbf{x}, \mathbf{w} \in \mathbb{R}^d$ and $\mathbf{A} \in \mathbb{R}^{d\times d}$. In the following questions, $\frac{\partial}{\partial \mathbf{x}}$ denotes the derivative with respect to $\mathbf{x}$, while $\nabla_{\mathbf{x}}$ denote the gradient with respect to $\mathbf{x}$. Compute the following:

(a) $\frac{\partial \mathbf{w}^T\mathbf{x}}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{w}^T\mathbf{x})$

(b) $\frac{\partial (\mathbf{w}^T\mathbf{A}\mathbf{x})}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{w}^T\mathbf{A}\mathbf{x})$

(c) $\frac{\partial (\mathbf{w}^T\mathbf{A}\mathbf{x})}{\partial \mathbf{w}}$ and $\nabla_{\mathbf{w}}(\mathbf{w}^T\mathbf{A}\mathbf{x})$

(d) $\frac{\partial (\mathbf{w}^T\mathbf{A}\mathbf{x})}{\partial \mathbf{A}}$ and $\nabla_{\mathbf{A}}(\mathbf{w}^T\mathbf{A}\mathbf{x})$

(e) $\frac{\partial (\mathbf{x}^T\mathbf{A}\mathbf{x})}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{x}^T\mathbf{A}\mathbf{x})$

(f) $\nabla^2_{\mathbf{x}}(\mathbf{x}^T\mathbf{A}\mathbf{x})$

2 Eigenvalues

(a) Let $\mathbf{A}$ be an invertible matrix. Show that if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then it is also an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue λ^{-1}.

(b) A square and symmetric matrix $\mathbf{A}$ is said to be positive semidefinite (PSD) ($\mathbf{A} \succeq 0$) if $\forall \mathbf{v} \neq 0, \mathbf{v}^T\mathbf{A}\mathbf{v} \geq 0$. Show that $\mathbf{A}$ is PSD if and only if all of its eigenvalues are nonnegative.

Hint: Use the eigendecomposition of the matrix $\mathbf{A}$.