

1 Vector Calculus

Below, $\mathbf{x} \in \mathbb{R}^d$ means that \mathbf{x} is a $d \times 1$ (column) vector with real-valued entries. Likewise, $\mathbf{A} \in \mathbb{R}^{d \times d}$ means that \mathbf{A} is a $d \times d$ matrix with real-valued entries. In this course, we will by convention consider vectors to be column vectors.

Consider $\mathbf{x}, \mathbf{w} \in \mathbb{R}^d$ and $\mathbf{A} \in \mathbb{R}^{d \times d}$. In the following questions, $\frac{\partial}{\partial \mathbf{x}}$ denotes the derivative with respect to \mathbf{x} , while $\nabla_{\mathbf{x}}$ denote the gradient with respect to \mathbf{x} . Compute the following:

- (a) $\frac{\partial \mathbf{w}^T \mathbf{x}}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{w}^T \mathbf{x})$
- (b) $\frac{\partial (\mathbf{w}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{w}^T \mathbf{A} \mathbf{x})$
- (c) $\frac{\partial (\mathbf{w}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{w}}$ and $\nabla_{\mathbf{w}}(\mathbf{w}^T \mathbf{A} \mathbf{x})$
- (d) $\frac{\partial (\mathbf{w}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{A}}$ and $\nabla_{\mathbf{A}}(\mathbf{w}^T \mathbf{A} \mathbf{x})$
- (e) $\frac{\partial (\mathbf{x}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{x}}$ and $\nabla_{\mathbf{x}}(\mathbf{x}^T \mathbf{A} \mathbf{x})$
- (f) $\nabla_{\mathbf{x}}^2(\mathbf{x}^T \mathbf{A} \mathbf{x})$

2 Eigenvalues

- (a) Let \mathbf{A} be an invertible matrix. Show that if \mathbf{v} is an eigenvector of \mathbf{A} with eigenvalue λ , then it is also an eigenvector of \mathbf{A}^{-1} with eigenvalue λ^{-1} .
- (b) A square and symmetric matrix \mathbf{A} is said to be positive semidefinite (PSD) ($\mathbf{A} \succeq 0$) if $\forall \mathbf{v} \neq 0, \mathbf{v}^T \mathbf{A} \mathbf{v} \geq 0$. Show that \mathbf{A} is PSD if and only if all of its eigenvalues are nonnegative.
Hint: Use the eigendecomposition of the matrix \mathbf{A} .