1 Unitary invariance

(a) Prove that the regular Euclidean norm (also called the $\ell^2$-norm) is unitary invariant; in other words, the $\ell^2$-norm of a vector is the same, regardless of how you apply a rigid linear transformation to the vector (i.e., rotate or reflect). Note that rigid linear transformation of a vector $v \in \mathbb{R}^d$ means multiplying by an orthogonal $U \in \mathbb{R}^{d \times d}$.

(b) Now show that the Frobenius norm of matrix $A$ is unitary invariant. The Frobenius norm is defined as $\|A\|_F = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = \sqrt{\text{tr}(A^T A)}$.

2 Vector Calculus

Below, $x \in \mathbb{R}^d$ means that $x$ is a $d \times 1$ (column) vector with real-valued entries. Likewise, $A \in \mathbb{R}^{d \times d}$ means that $A$ is a $d \times d$ matrix with real-valued entries. In this course, we will by convention consider vectors to be column vectors.

Consider $x, w \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d \times d}$. In the following questions, $\frac{\partial}{\partial x}$ denotes the derivative with respect to $x$, while $\nabla_x$ denote the gradient with respect to $x$. Compute the following:

(a) $\frac{\partial w^T x}{\partial x}$ and $\nabla_x (w^T x)$

(b) $\frac{\partial (w^T Ax)}{\partial x}$ and $\nabla_x (w^T Ax)$

(c) $\frac{\partial (w^T Ax)}{\partial w}$ and $\nabla_w (w^T Ax)$

(d) $\frac{\partial (w^T Ax)}{\partial A}$ and $\nabla_A (w^T Ax)$

(e) $\frac{\partial (x^T Ax)}{\partial x}$ and $\nabla_x (x^T Ax)$

(f) $\nabla^2_x (x^T Ax)$

3 Eigenvalues

(a) Let $A$ be an invertible matrix. Show that if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, then it is also an eigenvector of $A^{-1}$ with eigenvalue $\lambda^{-1}$.

(b) A square and symmetric matrix $A$ is said to be positive semidefinite (PSD) ($A \succeq 0$) if $\forall v \neq 0, v^T A v \geq 0$. Show that $A$ is PSD if and only if all of its eigenvalues are nonnegative.

Hint: Use the eigendecomposition of the matrix $A$. 