1 Probabilistic Graphical Models

Recall that we can represent joint probability distributions with directed acyclic graphs (DAGs). Let \( G \) be a DAG with vertices \( X_1, \ldots, X_k \). If \( P \) is a (joint) distribution for \( X_1, \ldots, X_k \) with (joint) probability mass function \( p \), we say that \( G \) represents \( P \) if

\[
p(x_1, \cdots, x_k) = \prod_{i=1}^{k} P(X_i = x_i | \text{pa}(X_i)),
\]

where \( \text{pa}(X_i) \) denotes the parent nodes of \( X_i \). (Recall that in a DAG, node \( Z \) is a parent of node \( X \) iff there is a directed edge going out of \( Z \) into \( X \)).

Consider the following DAG

![Figure 1: G, a DAG](image)

(a) Write down the joint factorization of \( P_{S,X,Y,Z}(s, x, y, z) \) implied by the DAG \( G \) shown in Figure 1.

(b) Is \( S \perp Z \mid Y \)?

(c) Is \( S \perp X \mid Y \)?
2 Hidden Markov Models: Math Review

A Hidden Markov Model is a Markov Process with unobserved (hidden) states.

Consider the following system in $\mathbb{R}^2$, where $X_n$ is the true state at any given time $n$ and $Y_n$ is our observation. Given an initial state $X_0$, we move to future states by recursively multiplying our current state with transformation matrix $A$ and adding i.i.d. Standard Normal Gaussian noise. When we take an observation $Y_n$ of the true state $X_n$, we are also exposed to i.i.d. Standard Normal Gaussian Noise.

$$X_{n+1} = AX_n + N(0, I)$$  \hspace{1cm} (2)  \\
$$Y_n = X_n + N(0, I)$$  \hspace{1cm} (3)

Where we have the 2x2 transformation matrix $A$ defined as follows:

$$A = \begin{bmatrix} .5 & -.25 \\ -.25 & .75 \end{bmatrix}$$  \hspace{1cm} (4)

If we restrict the initial state $X_0$ to be a unit vector ($\|X_0\|_2 = 1$), determine the following

(a) What are the eigenvalues of $A$? Is $A$ a positive semi-definite matrix? (Note that $\sqrt{5} = 2.236$)
(b) What is the $\| E[Y_{\infty}] \|_2$? Prove your assertion.

(c) Consider the Frobenius Norm of an arbitrary $M \times N$ matrix $Q$, defined as $\| Q \|_F = \sqrt{\sum_i \sum_j |Q_{i,j}|^2}$, which indicates the “magnitude” or “largeness” of a matrix. Is $\| Var[Y_{\infty}] \|_F$ finite or infinite? Prove your assertion.

You may find the following facts to be useful:

(i) Triangle Inequality: $\| X + Y \| \leq \| X \| + \| Y \|$

(ii) Cauchy Schwarz: $\| X Y \| \leq \| X \| \| Y \|$

(iii) Geometric Sum: $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$ \quad $\forall r$ s.t. $0 < r < 1$; $a, r \in \mathbb{R}$